TTIC 31260 Algorithmic Game Theory

01/28/26

Price of Anarchy, Price of Stability, Potential & Congestion Games

Your guide:

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[Readings: Ch. 17, 19.3 of AGT book]

High level

Now, switching to…

- Games with many players, but structured
  - Network routing, resource sharing,…
- Examining different questions
  - How much do we lose in terms of overall "quality" of the solution, if players are self-interested

General setup

n players. Player i chooses strategy $s_i \in S_i$.

- Overall state $s = (s_1, \ldots, s_n) \in S$.

  [Will only be considering pure strategies]

- Utility function $u_i: S \rightarrow \mathbb{R}$, or
- Cost function $cost_i: S \rightarrow \mathbb{R}$.
- (Sum) Social Welfare of s is sum of utilities over all players.
- If costs, called Sum Social Cost.
- Other things to care about: happiness of least-happy player, etc.

Price of Anarchy / Price of Stability

n players. Player i chooses strategy $s_i \in S_i$.

Say we're talking costs, so lower is better.

Price of Anarchy:

Ratio of cost of worst equilibrium to cost of social optimum. (worst-case over games in class)

Price of Stability:

Ratio of cost of best equilibrium to cost of social optimum. (worst-case over games in class)

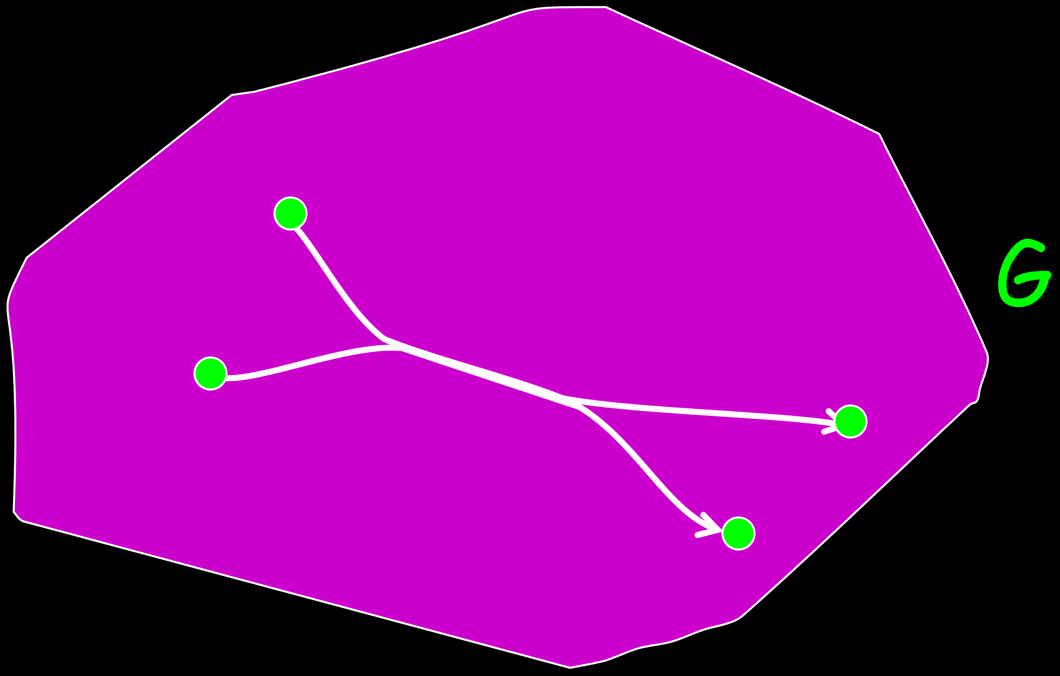
Example: Fair Cost-Sharing

- n players in weighted directed graph G.
- Player i wants to get from s_i to t_i.
- Each edge e has cost c_e.
- Players share the cost of edges they use with others using it.

Overloading s_i here – sorry.

This is what makes it a game

We will care about sum social cost



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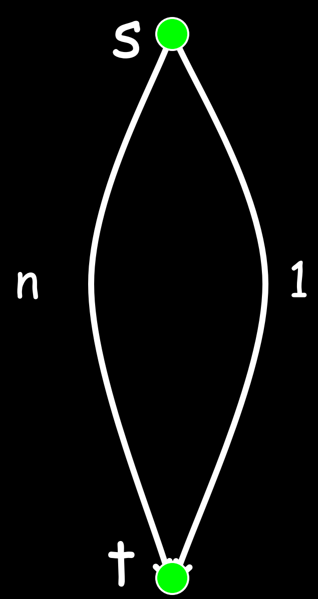
Also equilib



Social optimum: all use edge of cost 1.
(cost 1/n per player; total = 1)

Bad equilibrium: all use edge of cost n.
(cost 1 per player; total = n)

So, Price of Anarchy $\geq$ n.

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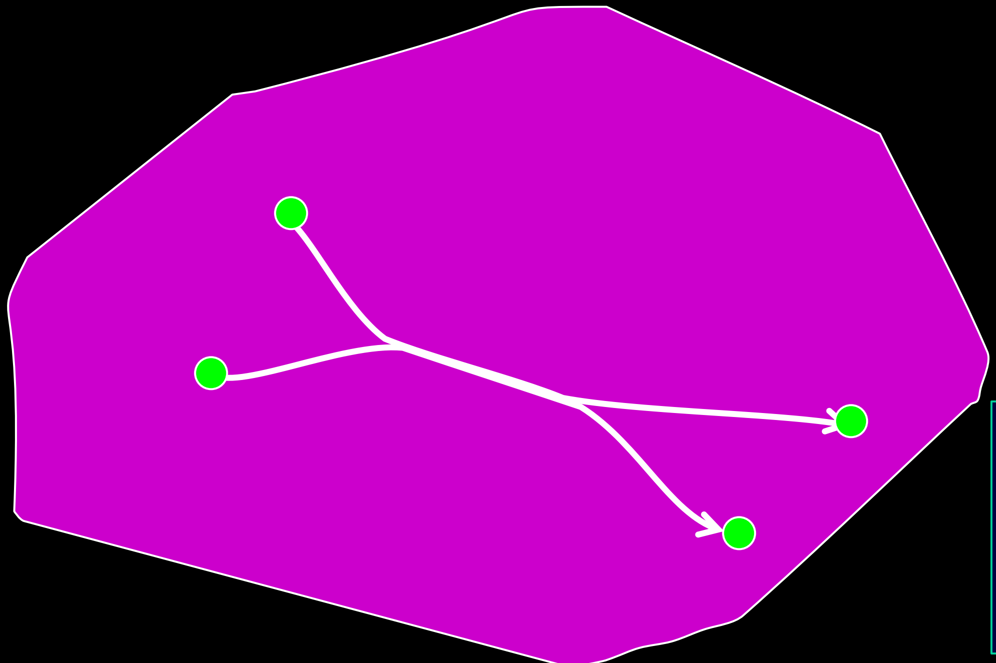
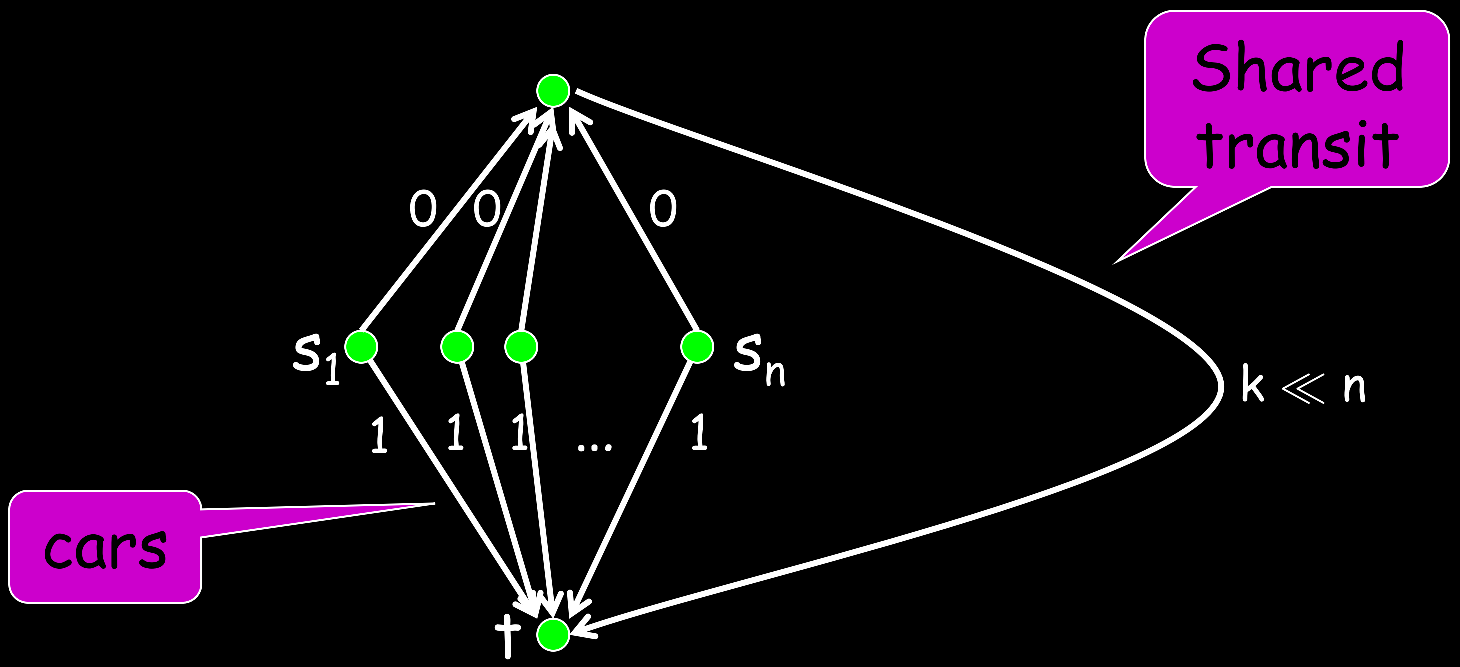
Can anyone see argument that Price of Anarchy $\leq n$?



- Cost(NE) $\leq \sum_i SP(s_i,t_i)$.
- Cost(OPT) $\geq \max_i SP(s_i,t_i)$.

Example: Fair Cost-Sharing

One more interesting example.



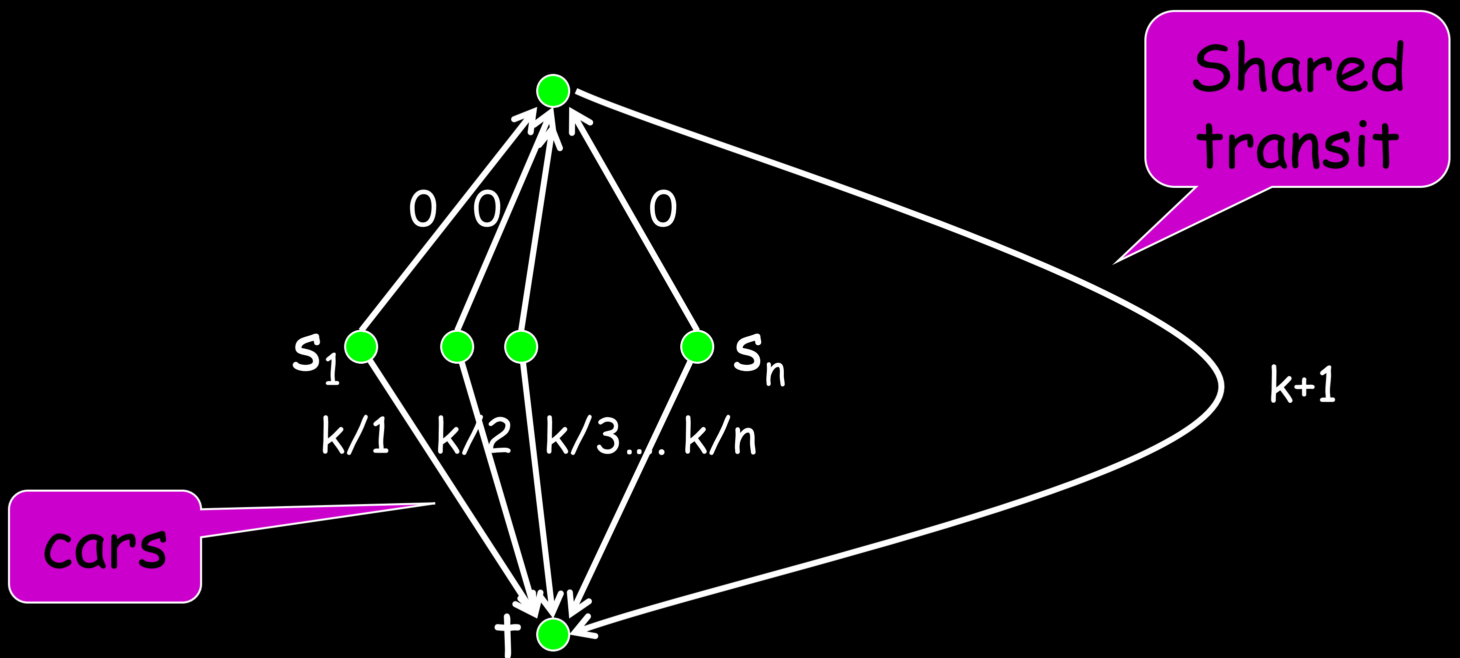
OPT has cost k (and is equilib). Also NE of cost n.

Now, let's modify it...

Example: Fair Cost-Sharing

One more interesting example.

Price of Stability
= $\Omega(\log n)$



OPT has cost k+1. Only equilib has cost k ln n.

Now, let's modify it…

Example: Fair Cost-Sharing

In fact, Price of Stability for fair cost-sharing is O(log n) too.

For this, we will use the fact that fair cost-sharing is an exact potential game…

Exact Potential Games

G is an exact potential game if there exists a function $\Phi(s)$ such that:

- For all players i, for all states $s = (s_i, s_{-i})$, for all possible moves to state $s' = (s_i', s_{-i})$,

$$\mathrm{cost}_i(s') - \mathrm{cost}_i(s) = \Phi(s') - \Phi(s)$$

- Notice that this implies there must exist a pure-strategy Nash equilibrium. Why?

- Furthermore, can reach by simple best-response dynamics. Each move is guaranteed to reduce the potential function.

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Claim: Fair cost-sharing is an exact potential game.

- Define potential $\Phi(s) = \sum_{e} \sum_{i=1}^{n_e(s)} c_e/i$

- If player changes from path p to path p′, pays $c_e/(n_e(s)+1)$ for each new edge, gets back $c_e/n_e(s)$ for each old edge. So, $\Delta\,\text{cost}_i = \Delta\,\Phi$.

Interesting fact about this potential

What is the gap between potential and cost?

$$\text{cost}(s) \le \Phi(s) \le \log(n) \times \text{cost}(s).$$

What does this imply about PoS?

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Interesting fact about this potential

What is the gap between potential and cost?

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What does this imply about PoS?

- Say we start at socially optimal state OPT.
- Do best-response dynamics from there until reach Nash equilibrium s.
- $\text{cost}(s) \leq \Phi(s) \leq \Phi(\text{OPT}) \leq \log(n) \times \text{cost}(\text{OPT})$.

So, Price of Stability = O(log n).

Fair cost-sharing summary

In every game:

- $\forall$ equilib s, cost(s) $\leq$ n $\times$ cost(OPT).
- $\exists$ equilib s, cost(s) $\leq$ log(n) $\times$ cost(OPT).

There exist games s.t.

- $\exists$ equilib s, cost(s) $\geq$ n $\times$ cost(OPT).
- $\forall$ equilib s, cost(s) $\geq$ clog(n) $\times$ cost(OPT).

Furthermore, potential function satisfies:

$$\text{cost}(s) \leq \Phi(s) \leq \log(n) \times \text{cost}(s).$$

So, starting from an arbitrary state, people optimizing for themselves can hurt overall cost but not too much.

Congestion Games more generally

Game defined by n players and m resources.

- Each player i choses a set of resources (e.g., a path) from collection S_i of allowable sets of resources (e.g., paths from s_i to t_i).
- Cost of resource j is a function $f_j(n_j)$ of the number n_j of players using it.
- Cost incurred by player i is the sum, over all resources being used, of the cost of the resource.
- Generic potential function: $$\sum_j \sum_{i=1}^{n_j} f_j(i)$$
- Best-response dynamics may take a long time to reach equilib, but if gap between Φ and cost is small, can get to apx-equilib fast.

Congestion Games & Potential Games

We just saw that every congestion game is an exact potential game.
[Rosenthal '73]

Turns out the converse is true as well.
[Monderer and Shapley '96]

For any exact potential game, can define resources to view it as a congestion game.

[see hwk]